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Minimal Coupling Method and the Dissipative Scalar Field Theory

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Quantum field theory of a damped vibrating string as the simplest dissipative scalar field is investigated by it's coupling to an infinite number of Klein–Gordon fields as the environment by introducing a minimal coupling method. Heisenberg equation containing a dissipative term proportional to the velocity is obtained for a special choice of coupling function and quantum dynamics for such a dissipative system is investigated. Some kinematical relations is calculated by tracing out the environment degrees of freedom. The rate of energy flowing between the system and it's environment is obtained.

KEY WORDS: minimal coupling method; dissipative scalar field theory; coupling function.

1. INTRODUCTION

In classical mechanics dissipation can be taken into account by introducing a velocity dependent damping term into the equation of motion. Such an approach is no longer possible in quantum mechanics where a time-independent Hamiltonian implies energy conservation and accordingly we can not find a unitary time evolution operator for both states and observable quantities consistently.

To investigate the quantum mechanical description of dissipative systems, there are some treatments, one can consider the interaction between two systems via an irreversible energy flow (Haken, 1975; Nicolis and Prigogine, 1977), or take a phenomenological treatment for a time dependent Hamiltonian which describes damped oscillations, here we can refer the interested reader to Caldirola–Kanai Hamiltonian for a damped harmonic oscillator (Caldirola, 1941).

$$H(t) = e^{-2\beta t}\frac{p^2}{2m} + e^{2\beta t}\frac{1}{2}m\omega^2 q^2 \tag{1}$$

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There are significant difficulties about the quantum mechanical solutions of the Caldirola–Kanai Hamiltonian, for example quantizing with this Hamiltonian violates the uncertainty relations or canonical commutation rules (Svinin, 1972) and the uncertainty relations vanish as time tends to infinity.

In 1931, Bateman (1931) presented the mirror-image Hamiltonian which consists of mirror Hamiltonians, one of them represents the main one-dimensional damped harmonic oscillator. Energy dissipated by the main oscillator completely will be absorbed by the other oscillator and thus the energy of the total system is conserved. Bateman hamiltonian is given by

$$H = \frac{p\bar{p}}{m} + \frac{\beta}{2m}(\bar{x}\bar{p} - xp) + \left(k - \frac{\beta^2}{4m}\right)x\bar{x}, \tag{2}$$

with the corresponding Lagrangian

$$L = m\dot{x}\dot{\bar{x}} + \frac{\beta}{2}(x\dot{\bar{x}} - \dot{x}\bar{x}) - kx\bar{x}, \tag{3}$$

canonical momenta for this dual system can be obtained from this Lagrangian as

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{\bar{x}} - \frac{\beta}{2}\bar{x}, \qquad \bar{p} = \frac{\partial L}{\partial \dot{\bar{x}}} = m\dot{x} + \frac{\beta}{2}x, \tag{4}$$

dynamical variables x, p and $\bar{p}$, $\bar{x}$ should satisfy the commutation relations

$$[x, p] = i, \qquad [\bar{x}, \bar{p}] = i, \tag{5}$$

however, the time-dependent uncertainty products obtained in this way, vanishes as time tends to infinity.

Caldirola (1941, 1983) developed a generalized quantum theory of a linear dissipative system in 1941: equation of motion of a single particle subjected to a generalized non conservative force Q can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q} + Q(q), \tag{6}$$

where $Q_r = -\beta(t)\frac{\partial T}{\partial \dot{q}_r} = -\beta(t)\sum a_{rj}\dot{q}_j$, and a_{rj}'s are some constants, changing the variable t to t^* using the following nonlinear transformation

$$t^* = \chi(t), \quad dt = \phi(t)\,dt^*, \quad \phi(t) = e^{-\int_0^t \beta(t')dt'}, \tag{7}$$

and from the definitions

$$\dot{q}^* = \frac{dq}{dt^*}, \quad L^* = L(q, \dot{q}^*, t^*), \quad p^* = \frac{\partial L^*}{\partial \dot{q}^*}, \tag{8}$$

the Lagragian equations (6) can be obtained from

$$\frac{d}{dt^*}\left(\frac{\partial L^*}{\partial \dot{q}^*}\right) - \frac{\partial L^*}{\partial q} = 0, \tag{9}$$

Canonical commutation rule and Schrodinger equation in this formalism are

$$[q, p^*] = i, \quad H^*\psi = i\frac{\partial \psi}{\partial t^*}, \tag{10}$$

where $H^* = \sum p^*\dot{q}^* - L^*$. But unfortunately uncertainty relations vanish as time goes to infinity.

Perhaps one of the effective approaches in quantum mechanics of dissipative systems is the idea of considering an environment coupled to the main system and doing calculations for the total system but at last for obtaining observables related to the main system, the environment degrees of freedom must be eliminated. The interested reader is referred to the Caldeira–Legget model (Caldeira and Legget, 1981, 1983). In this model the dissipative system is coupled with an environment made by a collection of N harmonic oscillators with masses m_n and frequencies ω_n, the interaction term in Hamiltonian is as follows

$$H' = -q\sum_{n=1}^{N} c_n x_n + q^2 \sum_{n=1}^{N} \frac{c_n^2}{2m_n\omega_n^2}, \tag{11}$$

where q and x_n denote coordinates of system and environment respectively and the constants c_n are called coupling constants.

The above coupling is not suitable for dissipative systems containing a dissipation term proportional to velocity. In fact with above coupling we can not obtain Heisenberg equation like $\ddot{q} + \omega^2 q + \beta\dot{q} = \xi(t)$ for a damped harmonic oscillator consistently and we can not study dissipative quantum fields, for example, a dissipative vibrating medium with this model. In this paper we generalize the Caldeira–Legget model to an environment with continuous degrees of freedom by a coupling similar to the coupling between a charged particle and the electromagnetic field known as minimal coupling. In Sections 2, the idea of a minimal coupling is introduced and the quantum dynamics of a damped vibrating string as the simplest scalar field theory, is investigated. In Section 3, quantum dynamics of the string and it's environment is investigated. In Section 4 some transition probabilities indicating the way dissipation flows, are obtained.

2. QUANTUM DYNAMICS OF A DAMPED VIBRATING STRING

In this section we consider a damped vibrating string as the dissipative system although the method is general and can be applied to a general scalar field. Quantum mechanics of a damped vibrating string with mass density λ, tension μ and length L, can be investigated by introducing a reservoir or an environment that interacts with the string through a new kind of minimal coupling. Let the two ends of the string be fixed in $x = 0$ and $x = L$ respectively and vibration be only in the y direction. If $\psi(x, t)$ is the wave function of the string, to quantizing $\psi(x, t)$, we assume $\psi(x, t)$ to be a hermitian operator and can be expanded in terms of orthogonal functions, $\sin\frac{n\pi x}{L}$ as follows

$$\psi(x,t)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{L\lambda\omega_n}}[a_n(t)+a_n^{\dagger}(t)]\sin\frac{n\pi x}{L}, \tag{12}$$

where $\omega_n=\sqrt{\frac{\mu}{\lambda}}\frac{n\pi}{L}$ and a_n and $a_n^{\dagger}$ are annihilation and creation operator of the string respectively and satisfy in any instance of time the following commutation rule

$$[a_n(t),a_m^{\dagger}(t)]=\delta_{nm}, \tag{13}$$

By definition of a conjugate canonical momentum density as

$$\pi_{\psi}(x,t)=i\sum_{n=1}^{\infty}\sqrt{\frac{\lambda\omega_n}{L}}[a_n^{\dagger}(t)-a_n(t)]\sin\frac{n\pi,x}{L} \tag{14}$$

then from (13) ψ,π_{ψ} satisfy equal time commutation relation

$$[\psi(x,t),\pi_{\psi}(x',t)]=i\delta(x-x'). \tag{15}$$

Hamiltonian of string is defined by

$$H_s=\int_0^L dx\left(\frac{\pi_{\psi}^2}{2\lambda}+\frac{1}{2}\mu\psi_x^2\right)=\sum_{n=1}^{\infty}\omega_n\left(a_n^{\dagger}a_n+\frac{1}{2}\right) \tag{16}$$

Let the total Hamiultonian, i.e., string plus environment be like this

$$H(t)=\int_0^L dx\frac{(\pi_{\psi}(x,t)-R(x,t))^2}{2\lambda}+\frac{1}{2}\mu\psi_x^2+H_B, \tag{17}$$

where ψ_x denotes derivative with respect to x and μ is a constant depending on string properties, H_B is the reservoir Hamiltonian

$$H_B(t)=\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}d^3k\omega_{\vec{k}}\left(b_{n\vec{k}}^{\dagger}(t)b_{n\vec{k}}(t)+\frac{1}{2}\right),\qquad \omega_{\vec{k}}=|\vec{k}|. \tag{18}$$

Annihilation and creation operators $b_{n\vec{k}}$, $b_{n\vec{k}}^{\dagger}$, in any instant of time, satisfy the following commutation relations

$$[b_{n\vec{k}}(t),b_{m\vec{k}'}^{\dagger}(t)]=\delta_{nm}\delta(\vec{k}-\vec{k}'), \tag{19}$$

and we will show later in Section 3 that reservoir is an infinite number of independent Klein–Gordon equations with a source term. Operator $R(x,t)$ have the basic role in interaction between string and reservoir and is defined by

$$R(x,t)=\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}d^3k[f(\omega_{\vec{k}})b_{n\vec{k}}(t)+f^*(\omega_{\vec{k}})b_{n\vec{k}}^{\dagger}(t)]\sin\frac{n\pi x}{L}, \tag{20}$$

let us call the function $f(\omega_k)$, the coupling function. Using (15), it can be shown easily that Heisenberg equation for $\psi(x,t)$ and $\pi_\psi(x,t)$ leads to

$$\begin{aligned}\dot{\psi}(x,t) &= i[H,\psi(x,t)] = \frac{\pi_\psi - R}{\lambda},\\ \dot{\pi}_\psi(x,t) &= i[H,\pi_\psi(x,t)] = \mu\psi_{xx},\end{aligned}\tag{21}$$

which after eleminating π_ψ, gives the following equation for the damped vibrating string

$$\lambda\ddot{\psi} - \mu\psi_{xx} = -\dot{R}(x,t).\tag{22}$$

Using (19) the Heisenberg equation for $b_{n\vec{k}}$, is

$$\dot{b}_{n\vec{k}} = i[H,b_{n\vec{k}}] = -i\omega_{\vec{k}}b_{n\vec{k}} + if^*(\omega_{\vec{k}})\int_0^L \dot{\psi}(x',t)\sin\frac{n\pi x'}{L}dx',\tag{23}$$

with the following formal solution

$$b_{n\vec{k}}(t) = b_{n\vec{k}}(0)e^{-i\omega_{\vec{k}}t} + if^*(\omega_{\vec{k}})\int_0^t dt' e^{-i\omega_{\vec{k}}(t-t')}\int_0^L \dot{\psi}(x',t')\sin\frac{n\pi x'}{L}dx',\tag{24}$$

substituting $b_{n\vec{k}}(t)$ from (24) into (22), using the relation $\delta(x-x') = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi x'}{L}$ and at last integrating over x' gives

$$\begin{aligned}&\lambda\ddot{\psi} - \mu\psi_{xx} + \int_0^t dt'\dot{\psi}(x,t')\gamma(t-t') = \xi(x,t),\\ &\xi(x,t) = i\int_{-\infty}^{+\infty} d^3k\omega_{\vec{k}}[f(\omega_{\vec{k}})b_{n\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - f^*(\omega_{\vec{k}})b^\dagger_{n\vec{k}}(0)e^{i\omega_{\vec{k}}t}]\sin\frac{n\pi x}{L},\\ &\gamma(t) = 4\pi L\int_0^\infty d\omega_{\vec{k}}|f(\omega_{\vec{k}})|^2\omega_{\vec{k}}^3\cos\omega_{\vec{k}}t,\end{aligned}\tag{25}$$

it is clear that the expectation value of $\xi(x,t)$ in any eigenstate of H_B, is zero. For the following special choice of coupling function

$$f(\omega_{\vec{k}}) = \sqrt{\frac{\beta}{4\pi^2 L\omega_{\vec{k}}^3}},\tag{26}$$

Equation (25) takes the form

$$\begin{aligned}&\lambda\ddot{\psi} - \mu\psi_{xx} + \beta\dot{\psi} = \tilde{\xi}(x,t),\\ &\tilde{\xi}(x,t) = i\sqrt{\frac{\beta}{4\pi^2 L}}\int_{-\infty}^{+\infty}\frac{d^3k}{\sqrt{\omega_{\vec{k}}}}[b_{n\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - b^\dagger_{n\vec{k}}(0)e^{i\omega_{\vec{k}}t}]\sin\frac{n\pi x}{L},\end{aligned}\tag{27}$$

Heisenberg equation for a_n and $a_n^\dagger$ is

$$\frac{\dot{a}_n + \dot{a}_n^\dagger}{\sqrt{L\lambda\omega_n}} = i\sqrt{\frac{\omega_n}{L\lambda}}(a_n^\dagger - a_n) - \frac{1}{\lambda}\int_{-\infty}^{+\infty} d^3k[f(\omega_{\vec{k}})b_{n\vec{k}}(t) + f^*(\omega_{\vec{k}})b_{n\vec{k}}^\dagger(t)],$$
$$i\sqrt{\frac{\lambda\omega_n}{L}}(\dot{a}_n^\dagger - \dot{a}_n) = -\lambda\omega_n^2\frac{(a_n^\dagger + a_n)}{\sqrt{\lambda L\omega_n}}. \tag{28}$$

Definition $A_n = \frac{a_n + a_n^\dagger}{\sqrt{L\lambda\omega_n}}$ and $B_n = i\sqrt{\frac{\lambda\omega_n}{L}}(a_n^\dagger - a_n)$ and using (26), we can easily obtain

$$\ddot{A}_n + \omega_n^2 A_n + \frac{\beta}{\lambda}\dot{A}_n = \zeta_n(t),$$
$$\zeta_n(t) = i\sqrt{\frac{\beta}{4\pi^2\lambda^2 L}}\int_{-\infty}^{+\infty}\frac{d^3k}{\sqrt{\omega_{\vec{k}}}}[b_{n\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - b_{n\vec{k}}^\dagger(0)e^{i\omega_{\vec{k}}t}], \tag{29}$$

with the following solution

$$A_n(t) = e^{-\frac{\beta t}{2\lambda}}(\hat{E}_n e^{i\Omega_n t} + \hat{F}_n e^{-i\Omega_n t}) + M_n(t),$$
$$M_n(t) = i\int_{-\infty}^{+\infty} d^3k\sqrt{\frac{\beta}{4\pi^2\lambda^2 L\omega_{\vec{k}}}}\left[\frac{b_{n\vec{k}}(0)}{\omega_n^2 - \omega_{\vec{k}}^2 - i\frac{\beta}{\lambda}\omega_{\vec{k}}}e^{-i\omega_{\vec{k}}t} - \frac{b_{n\vec{k}}^\dagger(0)}{\omega_n^2 - \omega_{\vec{k}}^2 + \frac{i\beta}{\lambda}\omega_{\vec{k}}}e^{i\omega_{\vec{k}}t}\right], \tag{30}$$

where $\Omega_n = \sqrt{\omega_n^2 - \frac{\beta^2}{4\lambda^2}}$. Operators $\hat{E}_n$ and $\hat{F}_n$, are specified by initial conditions

$$\hat{E}_n + \hat{F}_n = A_n(0) - M_n(0),$$
$$\left(\frac{-\beta}{2\lambda} + i\Omega_n\right)\hat{E}_n - \left(\frac{\beta}{2\lambda} + i\Omega_n\right)\hat{F}_n = \dot{A}_n(0) - \dot{M}_n(0), \tag{31}$$

solving above equations and substituting $\hat{E}_n$ and $\hat{F}_n$ in (30) one obtains

$$A_n(t) = e^{-\frac{\beta t}{2\lambda}}\left\{A_n(0)\cos\Omega_n t + \frac{\beta}{2\lambda\Omega_n}A_n(0)\sin\Omega_n t - \frac{\beta M_n(0)}{2\lambda\Omega_n}\sin\Omega_n t - M_n(0)\cos\Omega_n t + \frac{\dot{A}_n(0) - \dot{M}_n(0)}{\Omega_n}\sin\Omega_n t\right\} + M_n(t). \tag{32}$$

It is clear from (28) that $\dot{A}_n(0)$ is dependent on string and reservoir operators in $t = 0$. Substituting $\dot{A}_n(t)$ from (32) in (24) we can find a stable solution for $b_{n\vec{k}}(t)$

in $t \to \infty$ as

$$
\begin{aligned}
b_{n\vec{k}}(t) &= b_{n\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - i\sqrt{\frac{L\beta}{16\pi^2\omega_{\vec{k}}^3}}\frac{e^{-i\omega_{\vec{k}}t}}{(\omega_n^2-\omega_{\vec{k}}^2-\frac{i\beta}{\lambda}\omega_{\vec{k}})} \\
&\quad \left\{\omega_n^2 A_n(0) - \omega_n^2 M_n(0) + i\omega_{\vec{k}}(\dot{A}_n(0) - \dot{M}_n(0))\right\} \\
&\quad + \frac{i\beta}{8\pi^2\lambda\sqrt{\omega_{\vec{k}}^3}}\int_{-\infty}^{+\infty} d^3k'\sqrt{\omega_{\vec{k}'}}\left\{\frac{b_{n\vec{k}'}(0)}{\omega_n^2-\omega_{\vec{k}'}^2-\frac{i\beta}{\lambda}\omega_{\vec{k}'}}\frac{\sin\frac{(\omega_{\vec{k}}-\omega_{\vec{k}'}t)}{2}t}{\frac{(\omega_{\vec{k}}-\omega_{\vec{k}'})}{2}}e^{\frac{-i(\omega_{\vec{k}}+\omega_{\vec{k}'})t}{2}}\right. \\
&\quad \left. + \frac{b^\dagger_{n\vec{k}'}(0)}{\omega^2-\omega_{\vec{k}'}^2+\frac{i\beta}{m}\omega_{\vec{k}'}}\frac{\sin\frac{(\omega_{\vec{k}}+\omega_{\vec{k}'})}{2}t}{\frac{(\omega_{\vec{k}}+\omega_{\vec{k}'})}{2}}e^{\frac{i(\omega_{\vec{k}'}-\omega_{\vec{k}})t}{2}}\right\},
\end{aligned}
\tag{33}
$$

now substituting $b_{n\vec{k}}(t)$ from (33) in (28) and using (32), one can obtain operator $B_n(t)$

$$
B_n(t) = \lambda\dot{A}_n(t) + \sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d^3k\sqrt{\frac{\beta}{4\pi^2L\omega_{\vec{k}}^3}}[b_{n\vec{k}}(t) + b^\dagger_{n\vec{k}}(t)). \tag{34}
$$

A vector in fock space of string can be written like this

$$
|\Phi\rangle_s = \sum_{j=0}^{\infty}\sum_{n_1,\ldots,n_j=1}^{\infty}\Phi_{n_1,\ldots,n_j}|n_1\rangle_s\otimes\cdots\otimes|n_j\rangle_s \tag{35}
$$

where $|n\rangle_s$ denotes the state of a single particle in mode n, with corresponding wave function $\langle x|n\rangle = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$. Operators $a_n(0)$ and $a_n^\dagger(0)$ act on basis vectors $|n_1\rangle_s\otimes\cdots\otimes|n_j\rangle_s \equiv |n_1,\ldots,n_j\rangle_s$ as follows

$$
\begin{aligned}
a_m(0)|n_1,\ldots,n_j\rangle_s &= \sum_{r=1}^{j}\delta_{n_r,m}|n_1,\ldots,n_{r-1},n_{r+1},\ldots,n_j\rangle_s, \\
a_m^\dagger(0)|n_1,\ldots,n_j\rangle_s &= |m,n_1,\ldots,n_j\rangle_s,
\end{aligned}
\tag{36}
$$

also a vector in fock space of reservoir can be written as

$$
\begin{aligned}
|\Psi\rangle_B = \sum_{j=0}^{\infty}\sum_{\nu_1,\ldots,\nu_j=1}^{\infty}&\int d^3k_1\ldots d^3k_j\Psi_{n_1,\ldots,n_j}(\vec{k}_1,\ldots,\vec{k}_j) \\
&\times|\vec{k}_1,\nu_1\rangle_B\otimes\cdots\otimes|\vec{k}_j,\nu_j\rangle_B.
\end{aligned}
\tag{37}
$$

In subsequent section we show that reservoir is infinite number of independent Klein–Gordon fields and we can interpret $|\vec{k},\nu\rangle_B$ as a single particle state belong to νth Klein–Gordon field with corresponding momentum, $\vec{k}$. Operators $b_{n\vec{q}}(0)$ and

$b^{\dagger}_{n\vec{q}}(0)$ act on basis vectors $|\vec{k}_1, \nu_1\rangle_B \otimes \cdots \otimes |\vec{k}_j, \nu_j\rangle_B \equiv |\vec{k}_1, \nu_1, \ldots, \vec{k}_j, \nu_j\rangle_B$ as

$$\begin{aligned} & b_{n\vec{q}}(0)|\vec{k}_1, \nu_1, \ldots, \vec{k}_j, \nu_j\rangle_B \\ & \quad = \sum_{r=1}^{j} \delta_{n,\nu_r} \delta(\vec{q} - \vec{k}_r)|\vec{k}_1, \nu_1, \ldots \vec{k}_{r-1}, \nu_{r-1}, \vec{k}_{r+1}, \nu_{r+1}, \ldots, \vec{k}_j, \nu_j\rangle_B, \\ & b^{\dagger}_{n\vec{q}}(0)|\vec{k}_1, \nu_1, \ldots, \vec{k}_j, \nu_j\rangle_B = |\vec{q}, n, \vec{k}_1, \nu_1, \ldots, \vec{k}_j, \nu_j\rangle_B. \end{aligned} \tag{38}$$

If the state of system in $t = 0$ is taken to be $|\psi(0)\rangle = |0\rangle_B \otimes |m_1, \ldots m_r\rangle_s$ where $|0\rangle_B$ is vacuum state of reservoir and $|m_1, \ldots m_r\rangle_s$ an excited state of the Hamiltonian H_s then by making use of (32), (33), and (34) it can be shown that

$$\begin{aligned} & \lim_{t\to\infty}\Big[{}_B\langle 0| \otimes {}_s\langle m_1, \ldots, m_r| : \int_0^L dx \left[\frac{1}{2}\lambda\dot{\psi}^2 + \frac{1}{2}\mu\psi_x^2\right] \\ & \qquad : |m_1, \ldots, m_r\rangle_s \otimes |0\rangle_B\Big] = 0, \\ & \lim_{t\to\infty}\left[{}_B\langle 0| \otimes {}_s\langle m_1, \ldots, m_r| : \sum_{n=1}^{\infty} \omega_n a_n^{\dagger}(t) a_n(t) : |m_1, \ldots, m_r\rangle_s \otimes |0\rangle_B\right] \\ & = \lim_{t\to\infty}\left\{\sum_{n=1}^{\infty} \frac{L\beta^2\omega_n^4}{16\pi^2\lambda} \left|\int_{-\infty}^{+\infty} \frac{dx}{x} \frac{e^{ixt}}{\omega^2 - x^2 + i\frac{\beta}{\lambda}x}\right|^2 \right. \\ & \qquad \left. \times {}_s\langle m_r, \ldots m_1| : A_n^2(0) : |m_1, \ldots m_r\rangle_s \right\} \simeq \frac{\beta^2}{8\lambda^2} \sum_{i=1}^{r} \frac{1}{\omega_{m_i}}. \end{aligned} \tag{39}$$

where : : denotes the normal ordering operator. Now by substituting $b_{n\vec{k}}(t)$ from (33) into (18), we have

$$\begin{aligned} & \lim_{t\to\infty}\left[{}_B\langle 0| \otimes {}_s\langle m_r, \ldots, m_1| : \int d^3k \sum_{n=1}^{\infty} \omega_{\vec{k}} b^{\dagger}_{n\vec{k}} b_{n\vec{k}} : |m_1, \ldots m_r\rangle_s \otimes |0\rangle_B\right] \\ & \quad = \frac{\beta}{2\pi\lambda} \sum_{i=1}^{r} \omega_{m_i}^3 \int_0^{\infty} \frac{dx}{\left(\omega_{m_i}^2 - x^2\right)^2 + \frac{\beta^2}{\lambda^2}x^2}, \\ & \qquad + \frac{\beta}{2\pi\lambda} \sum_{i=1}^{r} \omega_{m_i} \int_0^{\infty} \frac{x^2 dx}{\left(\omega_{m_i}^2 - x^2\right)^2 + \frac{\beta^2}{\lambda^2}x^2}. \end{aligned} \tag{40}$$

3. QUANTUM FIELD OF RESERVOIR

Let us define the operators $Y_n(\vec{x},t)$ and $\Pi_n(\vec{x},t)$ as follows

$$Y_n(\vec{x},t)=\int_{-\infty}^{+\infty}\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\vec{k}}}}\left(b_{n\vec{k}}(t)e^{i\vec{k}.\vec{x}}+b^{\dagger}_{n\vec{k}}(t)e^{-i\vec{k}.\vec{x}}\right),$$

$$\Pi_n(\vec{x},t)=i\int_{-\infty}^{+\infty}d^3k\sqrt{\frac{\omega_{\vec{k}}}{2(2\pi)^3}}\left(b^{\dagger}_{n\vec{k}}(t)e^{-i\vec{k}.\vec{x}}-b_{n\vec{k}}(t)e^{i\vec{k}.\vec{x}}\right), \tag{41}$$

then using commutation relations (19), one can show that $Y_n(\vec{x},t)$ and $\Pi_m(\vec{x},t)$, satisfy the equal time commutation relations

$$[Y_n(\vec{x},t),\Pi_m(\vec{x}',t)]=i\delta_{nm}\delta(\vec{x}-\vec{x}'), \tag{42}$$

furthermore by substituting $b_{n\vec{k}}(t)$ from (24) in (41) we obtain

$$\frac{\partial\Pi_n(\vec{x},t)}{\partial t}=\nabla^2Y_n+L\dot{A}_n(t)P(\vec{x}),$$

$$P(\vec{x})=\mathrm{Re}\int_{-\infty}^{+\infty}d^3k\sqrt{\frac{\omega_{\vec{k}}}{2(2\pi)^3}}f(\omega_{\vec{k}})e^{-i\vec{k}.\vec{x}},$$

$$\Pi_n(\vec{x},t)=\frac{\partial Y_n}{\partial t}-L\dot{A}_n(t)Q(\vec{x}),\qquad Q(\vec{x})=\mathrm{Im}\int_{-\infty}^{+\infty}d^3k\frac{f(\omega_{\vec{k}})}{\sqrt{2(2\pi)^3\omega_{\vec{k}}}}e^{-i\vec{k}.\vec{x}}, \tag{43}$$

so for any n, $Y_n(\vec{x},t)$ satisfies the following source included Klein–Gordon equation

$$\frac{\partial^2Y_n}{\partial t^2}-\nabla^2Y_n=L\ddot{A}_n(t)Q(\vec{x})+2\dot{A}_n(t)P(\vec{x}), \tag{44}$$

with the corresponding Lagrangian density as follows

$$\pounds_n=\frac{1}{2}\left(\frac{\partial Y_n}{\partial t}\right)^2-\frac{1}{2}\vec{\nabla}Y_n.\vec{\nabla}Y_n-L\dot{A}_nQ(\vec{x})\frac{\partial Y_n}{\partial t}+L\dot{A}_nP(\vec{x})Y_n. \tag{45}$$

It is clear that the reservoir is made by an infinite number of massless Klein–Gordon fields containing the source term $2\ddot{A}_nQ(\vec{x})+2\dot{A}_nP(\vec{x})$. Hamiltonian density for (44) is

$$\aleph_n=\frac{(\Pi_n+L\dot{A}_nQ)^2}{2}+\frac{1}{2}|\vec{\nabla Y_n}|^2-L\dot{A}_nP(\vec{x},)Y_n, \tag{46}$$

and Equations (43) are Heisenberg equations for fields Y_n and Π_n. If we obtain $b_{n\vec{k}}$ and $b^{\dagger}_{n\vec{k}}$ from (41) in terms of Y_n and Π_n and substitute them in (18), we find

$$H_B=\int_{-\infty}^{+\infty}d^3k\omega_{\vec{k}}\left(b^{\dagger}_{n\vec{k}}b_{n\vec{k}}+\frac{1}{2}\right)=\frac{\Pi_n^2}{2}+\frac{1}{2}|\vec{\nabla Y_n}|^2. \tag{47}$$

4. TRANSITION PROBABILITIES

We can write the Hamiltonian (20) as

$$H = H_0 + H',$$
$$H_0 = H_s + H_B = \sum_{n=1}^{\infty}\left(a_n^\dagger a_n + \frac{1}{2}\right)\omega_n + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} d^3k\,\omega_{\vec{k}}\left(b_{n\vec{k}}^\dagger b_{n\vec{k}} + \frac{1}{2}\right)$$
$$H' = -\int_0^L dx\,\frac{\pi_\psi(x,t)}{\lambda}R(x,t) + \frac{R^2(x,t)}{2\lambda}, \tag{48}$$

and in interaction picture we can write

$$a_{nI}(t) = e^{iH_0t}a_n(0)e^{-iH_0t} = a_n(0)e^{-i\omega_n t},$$
$$b_{n\vec{k}I}(t) = e^{iH_0t}b_{n\vec{k}}(0)e^{-iH_0t} = b_{n\vec{k}}(0)e^{-i\omega_{\vec{k}}t}, \tag{49}$$

terms $\frac{R}{\lambda}\pi_\psi$ and $\frac{R^2}{2\lambda}$ are of the first order and second order of damping respectively, therefore for sufficiently weak damping, $\frac{R^2}{2\lambda}$ is small in comparison with $\frac{R}{\lambda}\pi_\psi$. Furthermore, $\frac{R^2}{2\lambda}$ has not any role in those transition probabilities where initial and final states of Hamiltonian of vibrating string are different, hence we can neglect the term $\frac{R^2}{2\lambda}$ and estimate H' by $-\frac{R}{\lambda}\pi_\psi$. Substituting a_{nI} and $b_{n\vec{k}I}$ from (49) into $-\frac{R}{\lambda}\pi_\psi$, one obtains H_I' in interaction picture, as

$$H_I' = -\frac{iL}{2\lambda}\sum_{n=1}^{\infty}\sqrt{\frac{\lambda\omega_n}{L}}\int_{-\infty}^{+\infty} d^3k\big(f(\omega_{\vec{k}})a_n^\dagger(0)b_{n\vec{k}}(0)e^{i(\omega_n-\omega_{\vec{k}})t}$$
$$+ f^*(\omega_{\vec{k}})a_n^\dagger(0)b_{n\vec{k}}^\dagger(0)e^{i(\omega_n+\omega_{\vec{k}})t} - f(\omega_k)a_n(0)b_{n\vec{k}}(0)e^{-i(\omega_{\vec{k}}+\omega_n)t}$$
$$-f^*(\omega_{\vec{k}})a_n(0)b_{n\vec{k}}^\dagger(0)e^{i(\omega_{\vec{k}}-\omega_n)t}\big), \tag{50}$$

terms containing just $a_n(0)b_{n\vec{k}}(0)$ and $a_n^\dagger(0)b_{n\vec{k}}^\dagger(0)$ violate the conservation of energy in the first order perturbation, because $a_n(0)b_{n\vec{k}}(0)$ destroys an excited state of string while at the same time destroying a reservoir excitation state and $a_n^\dagger(0)b_{n\vec{k}}^\dagger(0)$ creates an excited state of vibrating string, while creating an excited reservoir state at the same time, therefore we neglect the terms involving $a_n(0)b_{n\vec{k}}(0)$ and $a_n^\dagger(0)b_{n\vec{k}}^\dagger(0)$, because of energy conservation and estimate H_I' by

$$H_I' = -\frac{i}{2}\sqrt{\frac{L}{\lambda}}\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d^3k\sqrt{\omega_n}\big[f(\omega_{\vec{k}})a_n^\dagger(0)b_{n\vec{k}}(0)e^{i(\omega_n-\omega_{\vec{k}})t}$$
$$-f^*(\omega_{\vec{k}})a_n(0)b_{n\vec{k}}^\dagger(0)e^{-i(\omega_n-\omega_{\vec{k}})t}\big]. \tag{51}$$

The time evolution of density operator in interaction picture is as follows (Scully and Zubairy, 1997)

$$\rho_I(t) = U_I(t, t_0)\rho_I(t_0)U_I^\dagger(t, t_0), \tag{52}$$

where U_I is the time evolution operator, which in first order perturbation is

$$\begin{aligned} U_I(t, t_0 = 0) &= 1 - i\int_0^t dt_1 H_I'(t_1) \\ &= 1 - \frac{1}{2}\sqrt{\frac{L}{\lambda}}\int_{-\infty}^{+\infty} d^3k \sum_{n=1}^{\infty}\sqrt{\omega_n}\Big[f(\omega_{\vec{k}})a_n^\dagger(0)b_{n\vec{k}}(0)e^{\frac{i(\omega_n-\omega_{\vec{k}})t}{2}} \\ &\quad - f^*(\omega_{\vec{k}})a_n(0)b_{n\vec{k}}^\dagger(0)e^{\frac{-i(\omega_n-\omega_{\vec{k}})t}{2}}\Big]\frac{\sin\frac{(\omega_n-\omega_{\vec{k}})}{2}t}{\frac{(\omega_n-\omega_{\vec{k}})}{2}}. \end{aligned} \tag{53}$$

Let $\rho_I(0) = |m, \ldots, m\rangle_s^r \; {}_s^r\langle m, \ldots, m| \otimes |0\rangle_B \; {}_B\langle 0|$ where $|0\rangle_B$ is the vacuum state of reservoir and $|m, \ldots, m\rangle_s^r$ is an excited state of vibrating string, from now on by $|m, \ldots, m\rangle_s^r$, we mean a string state containing r phonons of mode m, substituting $U_I(t, 0)$ from (53) in (52) and taking trace over reservoir parameters, we obtain

$$\begin{aligned} \rho_{sI}(t) &:= \mathrm{Tr}_B(\rho_I(t)) = |m, \ldots, m\rangle_s^r \; {}_s^r\langle m, \ldots, m| \\ &\quad + \frac{rL\omega_m}{4\lambda}|m, \ldots, m\rangle_s^{r-1} \; {}_s^{r-1}\langle m, \ldots, m| \int_{-\infty}^{+\infty} d^3p |f(\omega_{\vec{p}})|^2 \frac{\sin^2\frac{(\omega_{\vec{p}}-\omega_m)}{2}t}{(\frac{\omega_{\vec{p}}-\omega_m}{2})^2}, \end{aligned} \tag{54}$$

where we have used the formula $\mathrm{Tr}_B[|\vec{k}, n\rangle_B \; {}_B\langle\vec{k}', s|] = \delta_{ns}\delta(\vec{k} - \vec{k}')$. In large time approximation, we can write $\sin^2\frac{(\omega_{\vec{p}}-\omega_m)}{2}t/(\frac{\omega_{\vec{p}}-\omega_m}{2})^2 = 2\pi t\delta(\omega_{\vec{p}} - \omega_m)$, which leads to the following relation for density matrix

$$\begin{aligned} \rho_{sI}(t) &= |m, \ldots, m\rangle_s^r \; {}_s^r\langle m, \ldots, m| + \frac{2L\pi^2\omega_m^3 rt|f(\omega_m)|^2}{\lambda} \\ &\quad \times |m, \ldots, m\rangle_s^{r-1} \; {}_s^{r-1}\langle m, \ldots, m|, \end{aligned} \tag{55}$$

from density matrix we can calculate the probability of transition $|m, \ldots, m\rangle_s^r \to |m, \ldots, m\rangle_s^{r-1}$ as

$$\begin{aligned} \Gamma_{|m,\ldots,m\rangle_s^r \to |m,\ldots,m\rangle_s^{r-1}} &= \mathrm{Tr}[|m, \ldots, m\rangle_s^{r-1} \; {}_s^{r-1}\langle m, \ldots, m|\rho(t)] \\ &= \mathrm{Tr}_s[|m, \ldots, m\rangle_s^{r-1} \; {}_s^{r-1}\langle m, \ldots, m|\rho_{sI}(t)] = \frac{2rL\pi^2\omega_m^3 rt|f(\omega_m)|^2}{\lambda}, \end{aligned} \tag{56}$$

where Tr_s means taking trace over string eigenstates. For the special choice (26), above transition probability becomes

$$\Gamma_{|m,\ldots,m\rangle_s^r \to |m,\ldots,m\rangle_s^{r-1}} = \frac{r\beta t}{2\lambda}, \tag{57}$$

which shows that the rate of phonon number reduction (energy flow), is constant. Now consider the case where the reservoir is an excited state in $t = 0$ for example $\rho_I(0) = |m, \cdots, m\rangle_s^r \; {}_s^r\langle m, \cdots, m| \otimes |1_{\nu_1,\vec{p}_1}, 1_{\nu_2,\vec{p}_2}, \ldots, 1_{\nu_j,\vec{p}_j}\rangle_B \; {}_B\langle 1_{\nu_1,\vec{p}_1}, 1_{\nu_2,\vec{p}_2}, \ldots, 1_{\nu_j,\vec{p}_j}|$ where $|\vec{p}_1, \nu_1 \ldots \vec{p}_j, \nu_j\rangle_B$ denotes a reservoir state containing j quanta with momenta $\vec{p}_1, \ldots, \vec{p}_j$ belonging to the ν_1,ν_2,...,ν_jth, Klein–Gordon field, respectively then by making use of

$$\begin{aligned}
&\mathrm{Tr}_B[b^\dagger_{n\vec{k}}|\vec{p}_1, \nu_1 \ldots \vec{p}_j, \nu_j\rangle_B \; {}_B\langle \vec{p}_1, \nu_1 \ldots \vec{p}_j, \nu_j|b_{m\vec{k}'}] = \delta_{nm}\delta(\vec{k} - \vec{k}'),\\
&\mathrm{Tr}_B[b_{n\vec{k}}|\vec{p}_1, \nu_1 \ldots \vec{p}_j, \nu_j\rangle_B \; {}_B\langle \vec{p}_1, \nu_1 \ldots \vec{p}_j, \nu_j|b^\dagger_{m\vec{k}'}]\\
&\quad = \sum_{l=1}^{j} \delta_{n,\nu_l}\delta_{m,\nu_l}\delta(\vec{k} - \vec{p}_l)\delta(\vec{k}' - \vec{p}_l),
\end{aligned} \tag{58}$$

for $m \neq \nu_1 \cdots \nu_j$, we find

$$\begin{aligned}
\rho_{sI}(t) &= |m, \cdots, m\rangle_s^r \; {}_s^r\langle m, \cdots, m| +\\
&+ \frac{rL\omega_m}{4\lambda}|m, \cdots, m\rangle_s^{r-1} \; {}_s^{r-1}\langle m, \cdots, m| \int_{-\infty}^{+\infty} d^3k|f(\omega_{\vec{k}})|^2 \frac{\sin^2 \frac{(\omega_{\vec{k}}-\omega_m)}{2}t}{\left(\frac{\omega_{\vec{k}}-\omega_m}{2}\right)^2}\\
&+ \frac{L}{4\lambda}\sum_{r=1}^{j} \omega_{\nu_r}|\nu_r\rangle_s \otimes |m, \cdots, m\rangle_s^r \; {}_s\langle \nu_r| \otimes \; {}_s^r\langle m, \cdots, m||f(\omega_{\vec{p}_r})|^2 \frac{\sin^2 \frac{(\omega_{\vec{p}_r}-\omega_{\nu_r})}{2}t}{\left(\frac{\omega_{\vec{p}_r}-\omega_{\nu_r}}{2}\right)^2}
\end{aligned} \tag{59}$$

which gives the transition probability for $|m, \ldots, m\rangle_s^r \to |m, \ldots, m\rangle_s^{r-1}$ and $|m, \ldots, m\rangle_s^r \to |\nu, m, \ldots, m\rangle_s^r$, respectively, as follows

$$\begin{aligned}
\Gamma_{|m,\ldots,m\rangle_s^r \to |m,\ldots,m\rangle_s^{r-1}} &= \frac{2\pi^2 L\omega_m^3 rt}{\lambda}|f(\omega_m)|^2,\\
\Gamma_{|m,\ldots,m\rangle_s^r \to |\nu\rangle_s \otimes |m,\ldots,m\rangle_s} &= \frac{\pi L\omega_\nu|f(\omega_\nu)|^2 t}{2\lambda}\sum_{r=1}^{j} \delta_{\nu,\nu_r}\delta(\omega_{\vec{p}_r} - \omega_\nu).
\end{aligned} \tag{60}$$

For the choice (26), we find

$$\Gamma_{|m,\cdots,m\rangle_s^r \to |m,\cdots,m\rangle_s^{r-1}} = \frac{r\beta t}{\lambda},$$

$$\Gamma_{|m,\cdots,m\rangle_s^r\to|\nu\rangle_s\otimes|m,\cdots,m\rangle_s}=\frac{\beta t}{4\pi\lambda\omega_\nu^2}\sum_{r=1}^{j}\delta_{\nu,\nu_r}\delta(\omega_{\vec{p}_r}-\omega_\nu). \tag{61}$$

Another important case is when the reservoir has a Maxwell–Boltzman distribution, so let $\rho_I(0)=|m,\ldots,m\rangle_s^r\ {}_s^r\langle m,\ldots,m|\otimes\rho_B^T$ where $\rho_B^T=e^{\frac{-H_B}{KT}}\big/TR_B(e^{\frac{-H_B}{KT}})$, then by making use of following relations

$$\begin{aligned}
&Tr_B\big[b_{n\vec{k}}\rho_B^T b_{m\vec{k}'}\big]=Tr_B[b^\dagger_{n\vec{k}}\rho_B^T b^\dagger_{m\vec{k}'}]=0,\\
&Tr_B\big[b_{n\vec{k}}\rho_B^T b^\dagger_{m\vec{k}'}\big]=\frac{\delta_{nm}\delta(\vec{k}-\vec{k}')}{e^{\frac{\omega_{\vec{k}}}{KT}}-1},\\
&Tr_B\big[b^\dagger_{n\vec{k}}\rho_B^T b_{m\vec{k}'}\big]=\delta_{nm}\delta(\vec{k}-\vec{k}')\frac{e^{\frac{\omega_{\vec{k}}}{KT}}}{e^{\frac{\omega_{\vec{k}}}{KT}}-1},
\end{aligned}\tag{62}$$

we can obtain the density operator $\rho_{sI}(t)$ in interaction picture as

$$\begin{aligned}
&\rho_{sI}(t):=Tr_B[\rho_I(t)]=|m,\cdots,m\rangle_s^r\ {}_s^r\langle m,\cdots,m|\\
&+\frac{L}{4\lambda}\sum_{n\neq m}\omega_n\{|n\rangle_s|m,\cdots,m\rangle_s^r\ {}_s\langle n|\ {}_s^r\langle m,\cdots,m|\\
&\times\int_{-\infty}^{+\infty}d^3k\frac{|f(\omega_{\vec{k}})|^2}{e^{\frac{\omega_{\vec{k}}}{KT}}-1}\frac{\sin^2\frac{(\omega_{\vec{k}}-\omega_n)}{2}t}{\left(\frac{\omega_{\vec{k}}-\omega_n}{2}\right)^2}\}\\
&+\frac{rL\omega_m}{4\lambda}|m,\cdots,m\rangle_s^{r-1}\ {}_s^{r-1}\langle m,\cdots,m|\int_{-\infty}^{+\infty}d^3k\frac{|f(\omega_{\vec{k}})|^2e^{\frac{\omega_{\vec{k}}}{KT}}}{e^{\frac{\omega_{\vec{k}}}{KT}}-1}\frac{\sin^2\frac{(\omega_{\vec{k}}-\omega_m)}{2}t}{\left(\frac{\omega_{\vec{k}}-\omega_m}{2}\right)^2}\\
&+\frac{(r+1)L\omega_m}{4\lambda}|m,\cdots,m\rangle_s^{r+1}\ {}_s^{r+1}\langle m,\cdots,m|\int_{-\infty}^{+\infty}d^3k\frac{|f(\omega_{\vec{k}})|^2}{e^{\frac{\omega_{\vec{k}}}{KT}}-1}\frac{\sin^2\frac{(\omega_{\vec{k}}-\omega_m)}{2}t}{\left(\frac{\omega_{\vec{k}}-\omega_m}{2}\right)^2},
\end{aligned}\tag{63}$$

which accordingly gives the following transition probabilities in long time approximation

$$\begin{aligned}
\Gamma_{|m,\cdots,m\rangle_s^r\to|m,\cdots,m\rangle_s^{r-1}}&=\frac{2\pi^2L\omega_m^3r|f(\omega_m)|^2t}{\lambda}\frac{e^{\frac{\omega_m}{KT}}}{e^{\frac{\omega_m}{KT}}-1},\\
\Gamma_{|m,\cdots,m\rangle_s^r\to|\nu\rangle_s\otimes|m,\cdots,m\rangle_s^r}&=\frac{2\pi^2L\omega_\nu^3t|f(\omega_\nu)|^2}{\lambda(e^{\frac{\omega_\nu}{KT}}-1)}\qquad \nu\neq m\\
\Gamma_{|m,\cdots,m\rangle_s^r\to|m,\cdots,m\rangle_s^{r+1}}&=\frac{(r+1)2\pi^2L\omega_m^3t|f(\omega_m)|^2}{\lambda(e^{\frac{\omega_m}{KT}}-1)}
\end{aligned}\tag{64}$$

substituting (26) in these recent relations, we find

$$\Gamma_{|m,\cdots,m\rangle_s^r \to |m,\cdots,m\rangle_s^{r-1}} = \frac{r\beta t}{\lambda} \frac{e^{\frac{\omega m}{KT}}}{e^{\frac{\omega m}{KT}} - 1},$$

$$\Gamma_{|m,\cdots,m\rangle_s^r \to |\nu\rangle_s \otimes |m,\cdots,m\rangle_s^r} = \frac{\beta t}{\lambda(e^{\frac{\omega_\nu}{KT}} - 1)} \qquad \nu \neq m$$

$$\Gamma_{|m,\cdots,m\rangle_s^r \to |m,\cdots,m\rangle_s^{r+1}} = \frac{(r+1)\beta t}{\lambda(e^{\frac{\omega m}{KT}} - 1)}. \tag{65}$$

So in very low temperatures the energy flows from oscillator to the reservoir by the rate $\frac{\beta}{2\lambda}$ and no energy flows from reservoir to oscillator.

5. CONCLUDING REMARKS

The Caldeira–Legget model generalized to the case where the environment has continuous degrees of freedom, for example, a Klein–Gordon field or an infinite number of Klein–Gordon fields. A minimal coupling method introduced which leads to a consistent investigation of the quantum dynamics of a large class of quantum dissipative systems. By choosing different coupling functions in (20), we could investigate another forms of dissipation. The rate of energy dissipation (energy flowing between the system and it's environment), was a constant. This problem can be extended to the case where the field R, becomes a general field for example a vector field, which is suitable for investigating three-dimensional quantum dissipative models.

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